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# COEFFICIENT BOUNDS FOR CERTAIN SUBCLASSES OF BI-PRESTARLIKE FUNCTIONS ASSOCIATED WITH THE GEGENBAUER POLYNOMIAL

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ABSTRACT. In this paper, we familiarize and explore a new subclass of bi-prestarlike functions defined in the open unit disk, related with Gegenbauer polynomials. Furthermore, we find estimates of first two coefficients of functions in these classes, making use of the Gegenbauer polynomials. Also, we obtain the Fekete-Szegö inequalities for function in these classes. Several consequences of the results are also pointed out as corollaries.

## 1. INTRODUCTION

Let $\mathcal{A}$ represent the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n \tag{1}$$

normalized by the conditions $f(0) = 0 = f'(0) - 1$ defined in the open unit disk

$$\triangle = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ comprising of functions of the form (1) which are also univalent in $\triangle$.

An analytic function $\varphi$ is subordinate to an analytic function $\psi$, written $\varphi(z) \prec \psi(z)$, provided there is an analytic function $\omega$ defined on $\triangle$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $\varphi(z) = \psi(\omega(z))$.

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the well-known subclasses of $\mathcal{S}$, consisting of starlike and convex functions of order $\alpha$, $0 \leq \alpha < 1$, respectively. The function

$$s(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \Psi_n(\alpha) z^n$$

where

$$\Psi_n(\alpha) = \left( \frac{\prod_{k=2}^{n} (k - 2\alpha)}{(n-1)!} \right) \tag{2}$$

is the well-known extremal function for the class $\mathcal{S}^*(\alpha)$. Also $f \in \mathcal{A}$ is said to be prestarlike functions of order $\alpha (0 \leq \alpha < 1)$, denoted by $\mathcal{R}(\alpha)$ if $f * s(z) \in \mathcal{S}^*(\alpha)$. We note that $\mathcal{R}(1/2) = \mathcal{S}^*(1/2)$ and $\mathcal{R}(0) = \mathcal{K}(0)$. Using the convolution techniques,

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**Dedicated to the memory of Professor Stephan Ruscheweyh 1944-2019.**

Ruscheweyh [16] introduced and studied the class of prestarlike functions of order $\alpha$.
For functions $f \in \mathcal{S}$, we have

$$f \in \mathcal{K}(0) \Longleftrightarrow zf' \in \mathcal{S}^*(0).$$

The Koebe one quarter theorem [3] ensures that the image of $\triangle$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying

$$f^{-1}(f(z)) = z, \ (z \in \triangle) \text{ and } f(f^{-1}(w)) = w \ (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\triangle$ if both $f$ and $f^{-1}$ are univalent in $\triangle$. Let $\Sigma$ denote the class of bi-univalent functions defined in the unit disk $\triangle$. Since $f \in \Sigma$ has the Maclaurian series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{3}$$

We notice that the class $\Sigma$ is not empty. For instance, the functions $z$, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of $\Sigma$. However, the Koebe function is not a member of $\Sigma$. Lately, Srivastava et al. [18] have essentially revived the study of analytic and bi-univalent functions, it was followed by such works as those by(see [1,2,4,12,13,15]). Several authors have introduced and examined subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [1,2,12,18]), bi-close-to-convex functions [5,11] . Recently, Jahangiri and Hamidi [8] introduced and studied certain subclasses of bi-prestarlike functions mentioned as below:

The expansion of $s(z) = \dfrac{z}{(1-z)^{2(1-\alpha)}}$ is given by

$$\begin{aligned} s(z) &= z + \frac{(2-2\alpha)}{1!}z^2 + \frac{(2-2\alpha)(3-2\alpha)}{2!}z^3 \\ &+ \frac{(2-2\alpha)(3-2\alpha)(4-2\alpha)}{3!}z^4 + \cdots . \end{aligned}$$

So, by the definition of Hadamard product, we have

$$\begin{aligned} F(z) &= \frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \\ &= s(z) * f(z) \\ &= z + \frac{(2-2\alpha)a_2}{1!}z^2 + \frac{(2-2\alpha)(3-2\alpha)a_3}{2!}z^3 \\ &+ \frac{(2-2\alpha)(3-2\alpha)(4-2\alpha)a_4}{3!}z^4 + \cdots \end{aligned}$$

equivalently

$$F(z) = z + \Psi_2(\alpha)a_2z^2 + \Psi_3(\alpha)a_3z^3 + \Psi_4(\alpha)a_4z^4 + \cdots \tag{4}$$

Similarly, for the inverse function $g = f^{-1}$, we note that $G(w) = F^{-1}(z)$ and is obtained as

$$G(w) = w - \frac{(2-2\alpha)a_2}{1!}w^2 + \frac{\left(4(2-2\alpha)^2a_2^2 - (2-2\alpha)(3-2\alpha)a_3\right)}{2!}w^3 + \cdots$$

equivalently

$$G(w) = w - \Psi_2(\alpha)a_2w^2 + \left(2\Psi_2^2(\alpha)a_2^2 - \Psi_3(\alpha)a_3\right)w^3 + \cdots . \tag{5}$$

In Geometric Function Theory, there have been many interesting and fruitful usages of a wide variety of special functions, $q-$ calculus and special polynomials (for example) the Fibonacci polynomials, Faber polynomials the Lucas polynomials, the Pell polynomials, the Pell-Lucas polynomials, and the Chebyshev polynomials of the second kind and Horadam polynomials are potentially important in a variety of disciplines in the mathematical, physical, statistical, and engineering sciences. These polynomials have been studied in several papers from a theoretical point of view and recently in case of bi-univalent functions (see [14, 19, 20] also the references cited therein). Here, in this article, we associate certain bi-univalent functions with Gegenbauer polynomials and then explores some properties of the class of bi-prestarlike functions based on earlier work of Jahangiri and Hamidi [8] .

We recall the Gegenbauer polynomials (for details see Kim et al., [9] and references cited therein) are given in terms of the Jacobi polynomials $P_n^{(\nu,\upsilon)}(x)$ with $\nu = \upsilon = \vartheta - \frac{1}{2}; (\vartheta > -\frac{1}{2}, \vartheta \neq 0)$ by

$$\begin{aligned}\mathfrak{U}_n^{\vartheta}(x) &= \frac{\Gamma(\vartheta+\frac{1}{2})\Gamma(n+2\vartheta)}{\Gamma(2\vartheta)\Gamma(n+\vartheta+\frac{1}{2})}P_n^{(\vartheta-\frac{1}{2},\vartheta-\frac{1}{2})}(x)\\ &= \binom{n+2\vartheta-1}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}(2\vartheta+n)_k}{(\vartheta+\frac{1}{2})_k}\left(\frac{x-1}{2}\right)^k\end{aligned} \tag{6}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ . From (6), we note that $\mathfrak{U}_n^{\vartheta}(x)$ is a polynomial of degree $n$ with real coefficients and $\mathfrak{U}_n^{\vartheta}(1) = \binom{n+2\vartheta-1}{n}$. The leading coefficient of $\mathfrak{U}_n^{\vartheta}(x)$ is $2^n\binom{n+\vartheta-1}{n}$. By the theory of Jacobi polynomials with $\mu = \upsilon = \vartheta - \frac{1}{2}, \vartheta > -\frac{1}{2},$ and $\vartheta \neq 0$, we get

$$\mathfrak{U}_n^{\vartheta}(-x) = (-1)^n\mathfrak{U}_n^{\vartheta}(x).$$

It is not difficult to show that $\mathfrak{U}_n^{\vartheta}(x)$ is a solution of the following Gegenbauer differential equation:

$$(1-x^2)y'' - (2\lambda)xy' + n(n+2\vartheta)y = 0.$$

The Rodrigues formula for the Gegenbauer polynomials is well known as the following:

$$(1-x^2)^{\vartheta-\frac{1}{2}}\mathfrak{U}_n^{\vartheta}(x) = \frac{(-2)^n(\vartheta)_n}{n!(n+2\vartheta)_n}\left(\frac{d}{dx}\right)^n(1-x^2)^{n+\vartheta-\frac{1}{2}}.$$

The above equation can be easily derived from the properties of Jacobi polynomials.

As is well known, the generating function of Gegenbauer polynomials is given by

$$\frac{2^{\vartheta-\frac{1}{2}}}{(1-2xt+t^2)^{\frac{1}{2}}(1-xt+\sqrt{1-2xt+t^2})^{\vartheta-\frac{1}{2}}} = \frac{(\vartheta-\frac{1}{2})_n}{(2\vartheta)_n}\mathfrak{U}_n^{\vartheta}(x)t^n. \tag{7}$$

This equation can be derived from the generating function of Jacobi polynomials.

From above equation (7), we note that

$$\Phi(t,x) = \frac{1}{(1–2xt+t^2)^{\vartheta}} = \sum_{n=0}^{\infty} \mathfrak{U}_n^{\vartheta}(x)t^n \qquad ; \quad (|t|<1, |x|\leq 1). \tag{8}$$

The proof of above is given in [21] and Kim et al., [9](also see [10]) extensively studied this results for different perspective. We note that , for $\vartheta = 1$; we get the Chebyshev Polynomials and $\vartheta = \frac{1}{2}$; we get the Legendre Polynomials. In 1935, Robertson [17] proved an integral representation for typically real valued $T_R$ functions,has the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are holomorphic in $\Delta$, real for $z \in (-1;1)$ and satisfy the condition

$$Imf(z)Imz > 0 \quad z \in \Delta \notin (-1;1)$$

namely $f \in T_R$ if and only if it has the representation

$$f(z) = \int_{-1}^{1} \frac{z}{1–2xz+z^2} d(\mu) \qquad ; \qquad z \in \Delta$$

where$\mu$ is a probability measure on $[-1,1]$. The notion of the class $T_R(\vartheta), \vartheta > 0$, has been extended in [22] to the class which is defined by the integral formula

$$f(z) = \int_{-1}^{1} \frac{z}{(1–2xz+z^2)^{\vartheta}} d(\mu) \qquad ; \qquad z \in \Delta \tag{9}$$

where $\mu$ is a probability measure on $[-1,1]$. Of course, we have $T_R(1) \equiv T_R$ and if $f$ given by(9) plays an important role in the geometric theory of holomorphic functions in the unit disk $\Delta$ then we have

$$a_n = \int_{-1}^{1} \mathfrak{U}_{n-1}^{\vartheta}(x) d(x)$$

where $\mathfrak{U}_n^{\vartheta}(x)$ is the Gegenbauer polynomial of degree $n$.
In particular,

$$\begin{aligned}
\mathfrak{U}_0^{\vartheta}(x) &= 1 \\
\mathfrak{U}_1^{\vartheta}(x) &= 2\vartheta x \qquad (10)\\
\mathfrak{U}_2^{\vartheta}(x) &= 2\vartheta(\vartheta+1)x^2 - \vartheta = 2(\vartheta)_2 x^2 - \vartheta \qquad (11)\\
3\mathfrak{U}_3^{\vartheta}(x)(x) &= 4\vartheta(\vartheta+1)(\vartheta+2)x^3 - 6\vartheta(\vartheta+1)x \\
&= 4(\vartheta)_3 x^3 - 6(\vartheta)_2 x \qquad (12)
\end{aligned}$$

where$(\vartheta)_n = \vartheta(\vartheta+1)(\vartheta+2)\cdots(\vartheta+n-1)$.

One can easily see that the class $T_R(\vartheta), \vartheta > 0$, is a compact and convex set in the linear space of holomorphic functions $f(z) = z + \sum_{n=2}^{\infty} a_n\ z^n$ which are holomorphic in $\Delta$, (which have the form endowed with the topology of local uniform convergence on compact subsets of $\Delta$. The importance of the class $T_R(\vartheta), \vartheta > 0$, follows as well from the paper of Hallen beck [7] who studied the extreme points of some families of univalent functions and proved that (co$\mathcal{A}$= closed convex hull of $\mathcal{A}$, ext $\mathcal{A}$= the set of the extremal points of $\mathcal{A}$): $co\mathcal{S}_R^*(1-\vartheta) = T_R(\vartheta)$ and $extco\mathcal{S}_R^*(1-\vartheta) = \{\frac{z}{(1–2xz+z^2)^{\vartheta}}; x \in [-1;1]\}$ denotes the class of holomorphic functions given below in (1) which are univalent and starlike of order$\alpha, \alpha \in [0;1)$ in $\Delta$ and have real coefficients. In this paper, motivated by recent works Jahangiri and Hamidi [8], we introduce a subclass bi-prestarlike function class associated with Gegenbauer polynomials and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the functions $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$ by subordination and consider the celebrated Fekete-Szegö problem. We also provide relevant connections of our results with those considered in earlier investigations.

Now we define a new subclass bi-prestarlike functions in the open unit disk, associated with Gegenbauer polynomials as below:
To start with for our discussions unless otherwise stated we let $0 \leq \lambda \leq 1, \vartheta > \frac{1}{2}$ and $x \in (\frac{1}{2}, 1]$.

**Definition 1.1.** For $0 \leq \lambda \leq 1$, and $x \in (\frac{1}{2}, 1]$ a function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$ if the following subordination hold:

$$(1-\lambda)\frac{zF'(z)}{F(z)} + \lambda\left(1 + \frac{zF''(z)}{F'(z)}\right) \prec \Phi(z, x) \tag{13}$$

and

$$(1-\lambda)\frac{wG'(w)}{G(w)} + \lambda\left(1 + \frac{wG''(w)}{G'(w)}\right) \prec \Phi(w, x) \tag{14}$$

where $z, w \in \Delta$ and $F$ and $G$ is given by (4) and (5), respectively.

**Remark 1.1.** Suppose $f \in \Sigma$. Then $\mathcal{R}_\Sigma(0, \alpha, \Phi(z, x)) \equiv \mathcal{PS}^*_\Sigma(\alpha, \Phi)$ :thus $f \in \mathcal{PS}^*_\Sigma(\alpha, \Phi)$ if the following subordination holds:

$$\frac{zF'(z)}{F(z)} \prec \Phi(z, x) \quad \textit{and} \quad \frac{wG'(w)}{G(w)} \prec \Phi(w, x)$$

where $z, w \in \Delta$ and $G$ is given by (5).

**Remark 1.2.** Suppose $f \in \Sigma$. Then $\mathcal{R}_\Sigma(1, \alpha, \Phi) \equiv \mathcal{K}^*_\Sigma(\alpha, \Phi)$ : thus $f \in \mathcal{K}^*_\Sigma(\alpha, \Phi)$ if the following subordination holds:

$$1 + \frac{zF''(z)}{F'(z)} \prec \Phi(z, x) \quad \textit{and} \quad 1 + \frac{wG''(w)}{G'(w)} \prec \Phi(w, x)$$

where $z, w \in \Delta$ and $g$ is given by (5).

## 2. Initial Taylor Coefficients $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$

**Theorem 2.1.** *Let $f$ given by (1) be in the class $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$. Then*

$$|a_2| \leq \frac{2\vartheta x\sqrt{2\vartheta x}}{\sqrt{|[2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)]4\vartheta^2x^2 - (2(\vartheta)_2x^2 - \vartheta)(1+\lambda)^2\Psi_2^2(\alpha)|}}. \tag{15}$$

*and*

$$|a_3| \leq \frac{4\vartheta^2x^2}{(1+\lambda)^2\Psi_2^2(\alpha)} + \frac{\vartheta x}{(1+2\lambda)\Psi_3(\alpha)}. \tag{16}$$

*where $0 \leq \lambda \leq 1$ and $x \neq \frac{(1+\lambda)\Psi_2(\alpha)}{2\sqrt{[(1+\lambda)\vartheta + 2(\vartheta+1)(1+\lambda)^2]\Psi_2^2(\alpha) - 2(1+2\lambda)\Psi_3(\alpha)}}$.*

*Proof.* Let $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi)$ and $g = f^{-1}$. Considering (13) and (14), we have

$$(1-\lambda)\frac{zF'(z)}{F(z)} + \lambda\left(1 + \frac{zF''(z)}{F'(z)}\right) = \Phi(z, x) \tag{17}$$

and

$$(1-\lambda)\frac{wG'(w)}{G(w)} + \lambda\left(1 + \frac{wG''(w)}{G'(w)}\right) = \Phi(w, x). \tag{18}$$

Define the functions $u(z)$ and $v(w)$ by

$$u(z) = c_1 z + c_2 z^2 + \cdots \tag{19}$$

and

$$v(w) = d_1 w + d_2 w^2 + \cdots \tag{20}$$

are analytic in $\triangle$ with $u(0) = 0 = v(0)$ and $|u(z)| < 1$, $|v(w)| < 1$, for all $z, w \in \triangle$. It is well-known that

$$|u(z)| = |c_1 z + c_2 z^2 + \cdots| < 1 \text{ and } |v(w)| = |d_1 w + d_2 w^2 + \cdots| < 1, z, w \in \triangle, \tag{21}$$

then

$$|c_j| \leq 1 \qquad \text{and} \qquad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}. \tag{22}$$

Using (19) and (20) in (17) and (18) respectively, we have

$$(1-\lambda)\frac{zF'(z)}{F(z)} + \lambda\left(1 + \frac{zF''(z)}{F'(z)}\right) = 1 + \mathfrak{U}_1^{\vartheta}(x)u(z) + \mathfrak{U}_2(x)u^2(z) + \cdots, \tag{23}$$

and

$$(1-\lambda)\frac{wG'(w)}{G(w)} + \lambda\left(1 + \frac{wG''(w)}{G'(w)}\right) = 1 + \mathfrak{U}_1^{\vartheta}(x)v(w) + \mathfrak{U}_2^{\vartheta}(x)v^2(w) + \cdots. \tag{24}$$

In light of (1) - (3), and from (23) and (24), we have

$$\begin{aligned} 1 + (1+\lambda)\Psi_2(\alpha)a_2 z + [2(1+2\lambda)\Psi_3(\alpha)a_3 - (1+3\lambda)\Psi_2^2(\alpha)a_2^2]z^2 + \cdots \\ = 1 + \mathfrak{U}_1^{\vartheta}(x)c_1 z + [\mathfrak{U}_1^{\vartheta}(x)c_2 + \mathfrak{U}_2^{\vartheta}(x)c_1^2]z^2 + \cdots, \end{aligned}$$

and

$$\begin{aligned} 1-(1+\lambda)\Psi_2(\alpha)a_2 w + \{[(8\lambda+4)\Psi_3(\alpha) - (3\lambda+1)\Psi_2^2(\alpha))]a_2^2 - 2(1+2\lambda)\Psi_3(\alpha)a_3\}w^2 + \cdots \\ = 1 + \mathfrak{U}_1^{\vartheta}(x)d_1 w + [\mathfrak{U}_1^{\vartheta}(x)d_2 + \mathfrak{U}_2^{\vartheta}(x)d_1^2]w^2 + \cdots. \end{aligned}$$

which yields the following relations:

$$(1+\lambda)\Psi_2(\alpha)a_2 = \mathfrak{U}_1^{\vartheta}(x)c_1, \tag{25}$$

$$-(1+3\lambda)\Psi_2^2(\alpha)a_2^2 + 2(1+2\lambda)\Psi_3(\alpha)a_3 = \mathfrak{U}_1^{\vartheta}(x)c_2 + \mathfrak{U}_2^{\vartheta}(x)c_1^2 \tag{26}$$

and

$$-(1+\lambda)\Psi_2(\alpha)a_2 = \mathfrak{U}_1^{\vartheta}(x)d_1, \tag{27}$$

$$(4(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha))a_2^2 - 2(1+2\lambda)\Psi_3(\alpha) = a_3 = \mathfrak{U}_1^{\vartheta}(x)d_2 + \mathfrak{U}_2^{\vartheta}(x)d_1^2 \tag{28}$$

From (25) and (27) it follows that

$$c_1 = -d_1 \tag{29}$$

and

$$2(1+\lambda)^2\Psi_2^2(\alpha)a_2^2 = [\mathfrak{U}_1^{\vartheta}(x)]^2(c_1^2 + d_1^2). \tag{30}$$

Adding (26) and (28), using (30), we obtain

$$a_2^2 = \frac{[\mathfrak{U}_1^{\vartheta}(x)]^3(c_2+d_2)}{2[\{2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)\}[\mathfrak{U}_1^{\vartheta}(x)]^2 - (1+\lambda)^2\Psi_2^2(\alpha)\mathfrak{U}_2^{\vartheta}(x)]}.$$

Applying (22) to the coefficients $c_2$ and $d_2$, and using (10); (11)we have

$$|a_2| \leq \frac{2\vartheta x\sqrt{2\vartheta x}}{\sqrt{|[2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)]4\vartheta^2 x^2 - (2(\vartheta)_2 x^2 - \vartheta)(1+\lambda)^2\Psi_2^2(\alpha)|}}. \tag{31}$$

By subtracting (28) from (26) and using (29),(30), we get

$$a_3 = \frac{[\mathfrak{U}_1^\vartheta(x)]^2(c_1^2+d_1^2)}{2(1+\lambda)^2\Psi_2^2(\alpha)} + \frac{\mathfrak{U}_1^\vartheta(c_2-d_2)}{4(1+2\lambda)\Psi_3(\alpha)}.$$

Using (10)and (11) , once again applying (22) to the coefficients $c_1, c_2, d_1$ and $d_2$, we get

$$|a_3| \le \frac{4\vartheta^2x^2}{(1+\lambda)^2\Psi_2^2(\alpha)} + \frac{\vartheta x}{(1+2\lambda)\Psi_3(\alpha)}. \tag{32}$$

□

By taking $\lambda = 0$ or $\lambda = 1$ and $x \in (0,1)$, one can easily state the estimates $|a_2|$ and $|a_3|$ for the function classes $\mathcal{R}_\Sigma(0,\alpha,\Phi) = \mathcal{PS}_\Sigma^*(\alpha,\Phi)$ and $\mathcal{R}_\Sigma(1,\alpha,\Phi) = \mathcal{K}_\Sigma^*(\alpha,\Phi)$ respectively.

**Remark 2.1.** Let $f$ given by (1) be in the class $\mathcal{PS}_\Sigma^*(\alpha,\Phi)$. Then

$$|a_2| \le \frac{2\vartheta x\sqrt{2\vartheta x}}{\sqrt{|[2\Psi_3(\alpha)-\Psi_2^2(\alpha)]4\vartheta^2x^2 - (2(\vartheta)_2x^2-\vartheta)\Psi_2^2(\alpha)|}}$$

and

$$|a_3| \le \frac{4\vartheta^2x^2}{\Psi_2^2(\alpha)} + \frac{\vartheta x}{\Psi_3(\alpha)}.$$

where $x \ne \frac{\Psi_2(\alpha)}{\sqrt{2[3\vartheta+1]\Psi_2^2(\alpha)-8\vartheta\Psi_3(\alpha)}}$.

**Remark 2.2.** Let $f$ given by (1) be in the class $\mathcal{K}_\Sigma^*(\alpha,\Phi)$. Then

$$|a_2| \le \frac{\vartheta x\sqrt{2\vartheta x}}{\sqrt{|[6\Psi_3(\alpha)-4\Psi_2^2(\alpha)]\vartheta^2x^2 - (2(\vartheta)_2x^2-\vartheta)\Psi_2^2(\alpha)|}}. \tag{33}$$

and

$$|a_3| \le \frac{\vartheta^2x^2}{\Psi_2^2(\alpha)} + \frac{\vartheta x}{3\Psi_3(\alpha)}. \tag{34}$$

where $x \ne \frac{\Psi_2(\alpha)}{\sqrt{2[3\vartheta+1]\Psi_2^2(\alpha)-6\vartheta\Psi_3(\alpha)}}$.

For $\alpha = 0$, Theorem 2.1 yields the following corollary.

**Corollary 2.1.** *Let $f$ given by (1) be in the class $\mathcal{R}_\Sigma(\lambda,0,\Phi) \equiv \mathcal{K}_\Sigma(\lambda,\Phi)$. Then*

$$|a_2| \le \frac{\vartheta x\sqrt{2\vartheta x}}{\sqrt{|2\vartheta^2x^2 - (2(\vartheta)_2x^2-\vartheta)(1+\lambda)^2|}} \tag{35}$$

*and*

$$|a_3| \le \frac{\vartheta^2x^2}{(1+\lambda)^2} + \frac{\vartheta x}{3(1+2\lambda)}. \tag{36}$$

*where $0 \le \lambda \le 1$ and $x \ne \frac{1+\lambda}{\sqrt{2(1+\lambda)^2(1+\vartheta)-2\vartheta}}$.*

By taking $\alpha = \frac{1}{2}$ in the above remarks we get the estimates $|a_2|$ and $|a_3|$ for the function classes $\mathcal{S}_\Sigma^*(\frac{1}{2},\Phi)$ and $\mathcal{K}_\Sigma^*(\frac{1}{2},\Phi)$.

**Remark 2.3.** Let $f$ given by (1) be in the class $\mathcal{S}_\Sigma^*(\frac{1}{2},\Phi)$. Then

$$|a_2| \le \frac{2\vartheta x\sqrt{2\vartheta x}}{\sqrt{|4\vartheta^2x^2 - 2(\vartheta)_2x^2+\vartheta|}}$$

and

$$|a_3| \le 4\vartheta^2x^2 + \vartheta x.$$

**Remark 2.4.** Let $f$ given by (1) be in the class $\mathcal{K}^*_\Sigma(\frac{1}{2\vartheta},\Phi)$. Then for $x \neq \frac{1}{\sqrt{2}}$,

$$|a_2| \leq \frac{\vartheta x\sqrt{2\vartheta x}}{\sqrt{|\vartheta - 2\vartheta^2 x^2|}}.$$

and

$$|a_3| \leq \vartheta^2 x^2 + \frac{\vartheta x}{3}.$$

## 3. Fekete-Szegö inequality for the function class $\mathcal{R}_\Sigma(\lambda,\alpha,\Phi)$

Due to Zaprawa [24], in this section we obtain the Fekete-Szegö inequality for the function classes $\mathcal{R}_\Sigma(\lambda,\alpha,\Phi)$.

**Theorem 3.1.** *Let $f$ given by (1) be in the class $\mathcal{R}_\Sigma(\lambda,\alpha,\Phi)$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\vartheta x}{(1+2\lambda)\Psi_3(\alpha)} \quad, |\mu - 1| \leq \frac{\left|\frac{(1+\lambda)^2}{4\vartheta^2 x^2}(2(\vartheta)_2 x^2 - \vartheta)\Psi_2^2(\alpha) + [\{2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)\}\right|}{2(1+2\lambda)\Psi_3(\alpha)} \\ \frac{8|1-\mu|\vartheta^3 x^3}{|[2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)]4\vartheta^2 x^2 - (2(\vartheta)_2 x^2 - \vartheta)(1+\lambda)^2\Psi_2^2(\alpha)|}, \\ \qquad |\mu - 1| \geq \frac{\left|\frac{(1+\lambda)^2}{4\vartheta^2 x^2}(2(\vartheta)_2 x^2 - \vartheta)\Psi_2^2(\alpha) + [\{2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)\}\right|}{2(1+2\lambda)\Psi_3(\alpha)} \end{cases}$$

**Proof.** From ((26)) and ((28))

$$a_3 - \mu a_2^2 = (1-\mu)\times \frac{[\mathfrak{U}_1^\vartheta(x)]^3(c_2+d_2)}{2[\{2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)\}[\mathfrak{U}_1^\vartheta(x)]^2 - (1+\lambda)^2\Psi_2^2(\alpha)\mathfrak{U}_2^\vartheta(x)]}. + \frac{\mathfrak{U}_1^\vartheta(x)(c_2-d_2)}{4(1+2\lambda)\Psi_3(\alpha)}$$

$$= \mathfrak{U}_1^\vartheta(x)\left[\left(h(\mu) + \frac{1}{4(1+2\lambda)\Psi_3(\alpha)}\right)c_2 + \left(h(\mu) - \frac{1}{4(1+2\lambda)\Psi_3(\alpha)}\right)d_2\right]$$

where

$$\begin{aligned} h(\mu) &= \frac{(1-\mu)[\mathfrak{U}_1^\vartheta(x)]^2}{2[\{2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)\}[\mathfrak{U}_1^\vartheta(x)]^2 - (1+\lambda)^2\Psi_2^2(\alpha)\mathfrak{U}_2^\vartheta(x)]} \\ &= \frac{(1-\mu)}{2[\{2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha)\} - \frac{(1+\lambda)^2}{4\vartheta^2 x^2}(2(\vartheta)_2 x^2 - \vartheta)\Psi_2^2(\alpha)} \end{aligned}$$

Then, in view of (10), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\vartheta x}{(1+2\lambda)\Psi_3(\alpha)}, & 0 \leq |h(\mu)| \leq \frac{1}{4(1+2\lambda)\Psi_3(\alpha)} \\ 4\vartheta x|h(\mu)|, & |h(\mu)| \geq \frac{1}{4(1+2\lambda)\Psi_3(\alpha)} \end{cases}$$

Taking $\mu = 1$, we have the following corollary.

**Corollary 3.1.** *If* $f \in \mathcal{R}_{\Sigma}(\lambda, \alpha, \Phi)$ *, then*

$$|a_3 - a_2^2| \le \frac{\vartheta x}{(1+2\lambda)\Psi_3(\alpha)}.$$

**Corollary 3.2.** *Let* $f$ *assumed as in* (1) *be in the class* $\mathcal{S}_{\Sigma}^{*}(\alpha, \Phi)$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{\vartheta x}{\Psi_3(\alpha)} \quad , \ |\mu - 1| \le \frac{\left|\frac{(2(\vartheta)_2 x^2 - \vartheta)}{4\vartheta^2 x^2}\Psi_2^2(\alpha) + [\{2\Psi_3(\alpha) - \Psi_2^2(\alpha)\}\right|}{2\Psi_3(\alpha)} \\ \frac{8|1-\mu|\vartheta^3 x^3}{|[2\Psi_3(\alpha) - \Psi_2^2(\alpha)]4\vartheta^2 x^2 - (2(\vartheta)_2 x^2 - \vartheta)\Psi_2^2(\alpha)|}, \\ \qquad |\mu - 1| \ge \frac{\left|\frac{(2(\vartheta)_2 x^2 - \vartheta)}{4\vartheta^2 x^2}\Psi_2^2(\alpha) + [\{2\Psi_3(\alpha) - \Psi_2^2(\alpha)\}\right|}{2\Psi_3(\alpha)} \end{cases}$$

*Especially, for* $\mu = 1$ *if* $f \in \mathcal{S}_{\Sigma}^{*}(\frac{1}{2}, \Phi(z, x))$ *we obtain*

$$|a_3 - a_2^2| \le \ \vartheta x.$$

**Corollary 3.3.** *Let* $f$ *assumed as in* (1) *be in the class* $\mathcal{K}_{\Sigma}^{*}(\alpha, \Phi)$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{\vartheta x}{3\Psi_3(\alpha)} \quad , \ |\mu - 1| \le \frac{\left|\frac{(2(\vartheta)_2 x^2 - \vartheta)}{\vartheta^2 x^2}\Psi_2^2(\alpha) + [\{6\Psi_3(\alpha) - 4\Psi_2^2(\alpha)\}\right|}{6\Psi_3(\alpha)} \\ \frac{2|1-\mu|\vartheta^3 x^3}{|[6\Psi_3(\alpha) - 4\Psi_2^2(\alpha)]\vartheta^2 x^2 - (2(\vartheta)_2 x^2 - \vartheta)\Psi_2^2(\alpha)|}, \\ \qquad |\mu - 1| \ge \frac{\left|\frac{(2(\vartheta)_2 x^2 - \vartheta)}{\vartheta^2 x^2}\Psi_2^2(\alpha) + [\{6\Psi_3(\alpha) - 4\Psi_2^2(\alpha)\}\right|}{6\Psi_3(\alpha)} \end{cases}$$

*Especially, for* $\mu = 1$ *if* $f \in \mathcal{K}_{\Sigma}^{*}(\frac{1}{2}, \Phi)$ *we obtain*

$$|a_3 - a_2^2| \le \frac{\vartheta x}{3}.$$

## 4. Conclusion

By fixing $\vartheta = \frac{1}{2}$ one can get the new analogues results for the subclasses discussed in this article based on Legendre polynomials further by taking $\vartheta = 1$ we get the results related with Chebyshev polynomials,discussed by Güney et al., [6].

## 5. Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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